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Prediction of elastic moduli of solids with oriented porosity

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There have been no simple equations available to predict the effects of arbitrarily shaped voids over the entire range of porosity encountered in real materials. Empirical expressions have been proposed, but they agree neither with appropriate theoretical analyses nor with extensive experimental data. Theoretical predictions have been linear in porosity and thus predict an insufficient reduction. Only a few analyses account for void shapes other than spherical. The present work represents a semi-empirical approach to fill these information deficiences.

1. Introduction

Porosity has been a continuing problem, with regard to both strength and modulus, for many classes of materials, especially ceramics and rocks. There have been increasing demands placed upon materials specialists to develop an ever-growing variety of new materials and their associated processes. As a result, not only have materials with intentional porosity been developed, such as syntactic foam for submersibles and those for orthopaedic uses, but also many processes used to manufacture filamentary composite materials result in high volume fractions of highly directional voids. Since it is costly and time consuming to test all configurations and combinations of materials, it is highly desirable to have mathematical expressions for predicting the effects of different kinds and amounts of porosity on Young's modulus.

Empirical predictions of the effect on strength [1], Young's modulus [2], and shear modulus [3] have taken the form of a simple exponential. For example, for Young's modulus [2] (E):

$$E/E_0 = e^{-b_E V_p} \tag{1}$$

where E_0 is the Young's modulus of the fully densified material, V_p is the porosity (volume fraction of pores), e is the base of the natural logarithms, and b_E is a constant determined empirically from experimental data.

Another form of empirical expression was proposed by McAdam [4]:

$$E/E_0 = (1-V_p)^m \tag{2}$$

where the only empirical constant is m.

The results of various theoretical analyses of the stresses induced in an isolated spherical cavity in an infinite medium subjected to all-around spherical tension and to pure shear have been applied to the prediction of the effect of spherical porosity on the bulk and shear moduli, respectively. Apparently, the most accurate of these analyses are those of Mackenzie [5] and of Hashin [6]. Their expressions for bulk modulus are identical and exact, while their shear-modulus expressions differ somewhat.

In an attempt to obtain better estimates of the Young's modulus at relatively high inclusion volume fractions, Paul [7] applied the Voigt model (rule of mixtures) to a cubic shaped inclusion. By setting the properties of the inclusion material equal to zero to cover voids, one can reduce Paul's equation to the following:

$$E/E_0 = (1-V_p^{2/3})[1-(1-V_p^{1/3})V_p^{2/3}]^{-1} \tag{3}$$

Later Ishai and Cohen [8] applied the Reuss model (inverse rule of mixtures) to a solid containing a cubic shaped inclusion or void. For the

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case of voids, their equation is:

$$E/E_0 = 1 - V_p^{2/3} \quad (4)$$

The first rigorous analysis of the elastic moduli of solids containing isolated nonspherical voids was carried out by Wu [9]. He treated a general spherical shape of cavity or inclusion, and presented quantitative results for three limiting geometrical shapes: disc, needle, and sphere. However, in all cases, the slope of E/E_0 against V_p was constant and thus unrealistic, except at very low porosities.

Janowski and Rossi [10] noted that the initial slope of the E/E_0 against V_p curve is an index of pore shape, which they attributed to stress concentration. This hypothesis was pursued in detail by Rossi [11], who claimed that the initial slope, $dR(0)/dV_p$, of the $R(\equiv E/E_0)$ against V_p curve is exactly equal to the negative of the stress concentration factor*, K_0, for an isolated pore ($V_p = 0$), i.e.

$$dR(0)/dV_p = -K_0 \quad (5)$$

Rossi checked his hypothesis by noting that it gives the correct theoretical values of $dR(0)/dV_p$ for a spherical void (-2) and a cylindrical void oriented parallel to the loading direction (-1)† Then Rossi applied his hypothesis to an axisymmetric ellipsoidal void by curve-fitting a simple hyperbolic curve to the theoretical results of Edwards [12]. The resulting expression was as follows:

$$R = 1 - (3 + 5a/c)(V_p/4) \quad (6)$$

where c/a is the aspect ratio of the void (ratio of the void length parallel to the loading direction to its width perpendicular to the loading direction). As can be seen, Equation 6 is still linear in V_p and thus valid only for small values of V_p.

Rossi's work is the departure point for the present work. We attempted to generalize his work in two ways: (1) extend Equation 5 to arbitrary values of V_p, and (2) a simple empirical equation is developed which is in agreement with Equation 5 for small values of V_p and which fits available experimental and theoretical data at large V_p values.

2. Checks on Rossi's hypothesis and on a possible generalization of it

First we verify the original Rossi hypothesis by comparison with experimental data and real materials. For spherical voids, the most accurate data are those reported by Hasselman and Fulrath [13] for artificial spherical voids. The slope $dR/dV_p = -2$ is in agreement with Rossi's hypothesis, but data were taken for V_p up to 0.024 only. For cylindrical voids oriented perpendicular to the direction of loading, the curve presented by Hasselman and Fulrath [13], based on considerable test data analysed statistically by Knudsen [14], is appropriate. Again the initial slope of -3 is in agreement with Rossi's hypothesis.

In the curve of Hasselman and Fulrath [13], as well as the experimental spherical-porosity curve presented by Ishai and Cohen [8] for V_p up to 0.70, the slope dR/dV_p decreases in magnitude as V_p is increased (see the experimental curves in Figs. 1 and 2). Also, it is well known, from the theory of interacting free-surface stress concentrators, i.e. including voids but not inclusions, that as multiple stress concentrators are moved closer together, their interaction causes a reduction in stress concentration factor (K). Thus, as V_p is increased, the voids become closer together and K would be expected to decrease. This suggests the following generalization of Rossi's hypothesis to arbitrary volume fractions:

$$dR/dV_p = -K \quad (7)$$

The "initial conditions" of the R against V_p curve are as follows:

$$R(0) = 1 \quad (8)$$

$$dR(0)/dV_p = -K_0 \text{ (Rossi's hypothesis)} \quad (9)$$

Since it is virtually impossible to measure the maximum stress at a void in a multi-void material, theoretical analyses must be relied upon to check the validity of Equation 5. Owing to the three-dimensional geometric complexity of theoretical analysis of a medium containing multiple spherical voids, no such analysis has yet appeared, although it is theoretically possible with the finite-element methods; see Agarwal *et al.* [15]. Thus, one must resort to cylindrical voids as the check cases.

For any system of parallel, prismatic (cylindrical) voids of arbitrary cross-sectional shape or combinations of shapes, loaded in the direction parallel to their axes, the reduction in stiffness is related only to the decrease in actual cross-sectional area. Thus, the stiffness ratio can be

*The stress concentration factor is defined as the maximum stress reached in the body divided by the applied field stress.

†It also checks out correctly for a cylindrical void oriented perpendicular to the loading direction (-3).

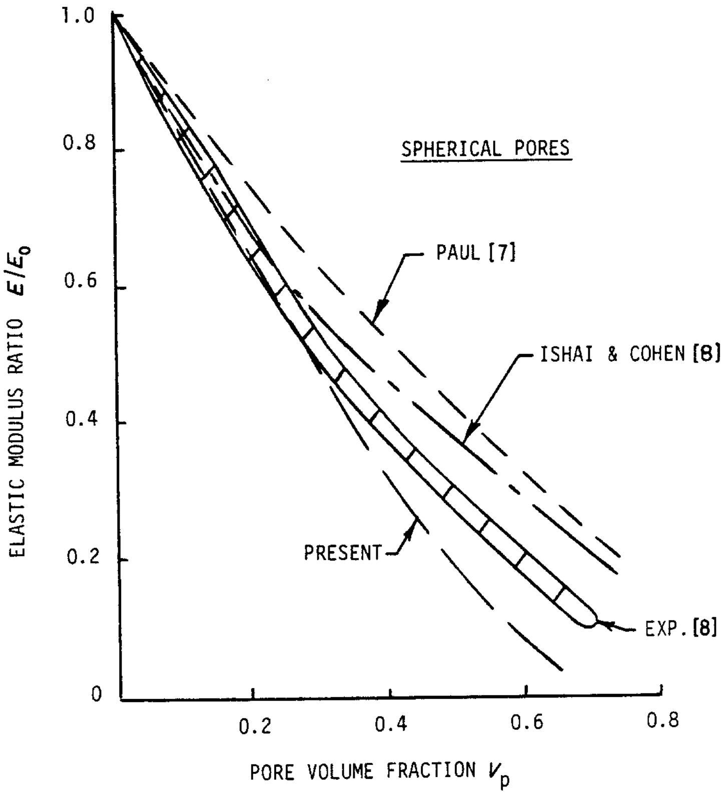

Figure 1 Effect of pore volume fraction on relative elastic modulus for a solid containing spherical voids.

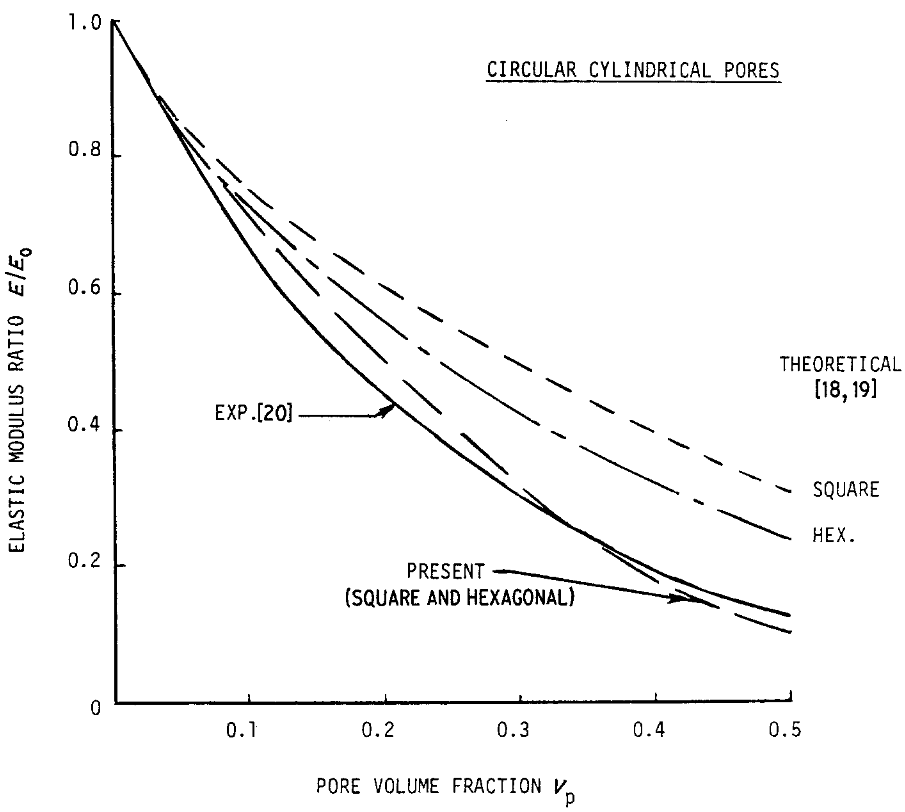

Figure 2 Effect of pore volume fraction on relative elastic modulus for a solid containing circular cylindrical voids.

TABLE I Comparisons of gross-section stress concentration factors with the slopes of the curves of relative modulus versus porosity for two arrays of circular cylindrical pores

Hexagonal array			Square array		
V_p	SCF_{gross} [18]	$-dR/dV_p$	V_p	SCF_{gross} [18]	$-dR/dV_p$
0.036	2.978	2.63	0.031	2.91	2.63
0.1451	2.830	1.80	0.126	2.72	1.76
0.3265	2.438	1.27	0.283	2.59	1.39
0.5804	2.329	0.65	0.502	2.88	0.99
0.7345	2.840	0.46	0.635	3.85	1.00

expressed as follows:

$$R \equiv E/E_0 = 1 - V_p \tag{10}$$

Since there is no stress concentration for the above case, i.e. $K = 1$ independent of V_p, it is seen that the generalized hypothesis is valid in this case.

Another case of cylindrical voids for which theoretical solutions are available is the case of a medium containing parallel, circular-cross-section cylindrical voids and loaded perpendicular to the axes of the voids [16–19]. In this case, the modulus depends upon the array (i.e. geometrical arrangement). Fig. 2 shows the results of theoretical analyses for the two most common arrays: square and equilateral triangular (hexagonal). Table I shows a comparison of the slopes of the curves with the corresponding theoretical stress concentration factors at the same porosity.

From Table I, it is clear that the Rossi hypothesis *cannot* be generalized from its original form, Equation 5, to Equation 7. Because of this, a new, simple empirical equation has been designed which is in agreement with the original Rossi hypothesis and all known theoretical solutions for small V_p and which gives a good fit to available experimental and theoretical data at large values of V_p.

3. A new semi-empirical relation for Young's moduli of porous media

The criteria for a good equation to predict E/E_0 against V_p are:

1. it should agree with known analytical solutions for $V_p \to 0$, i.e. it should satisfy the original Rossi equation, Equation 5;
2. it should agree with available test data for large values of V_p;
3. it should agree with known numerical analysis for large values of V_p;
4. it should be as simple as possible to permit computation by hand-held calculator or microcomputer, rather than by large-scale computer.

An equation which appears to come closest to meeting the above criteria is as follows:

$$R \equiv E/E_0 = [1 - (V_p/V_{p,max})]^{K_0 V_{p,max}} \tag{11}$$

where K_0 is the stress concentration factor for an isolated void (i.e. at $V_p = 0$) of the class under consideration, and $V_{p,max}$ is the maximum porosity geometrically possible. Table II lists the values of K_0 and $V_{p,max}$ for various void geometries.

The equation referred to in Table II is from Rossi [11]:

TABLE II Initial stress concentration factors and maximum geometrically possible porosity for various void geometries and directions of loading

Void geometry	Loading direction	K_0	$V_{p,max}$
Cylindrical, any cross-sectional shape	Parallel to axis	1	1
Circular cylindrical hexagonal array	Perpendicular to axis	3	$(\pi/6)3^{1/2} \approx 0.9069$
Circular cylindrical square array	Perpendicular to axis	3	$\pi/4 \approx 0.7854$
Spherical, hexagonal close-packed array	Any direction	2	$(\pi/4)2^{1/2} \approx 0.7405$
Spheroid of revolution square prismatic array	Parallel to axis	See Equation 12	$\pi/6 \approx 0.5236$
Spheroid of revolution, hexagonal prismatic array	Parallel to axis	See Equation 12	$(\pi/9)3^{1/2} \approx 0.6046$

$$K_0 = 0.75 + (1.25a/c) \quad (12)$$

4. Conclusion

A new semi-empirical equation, Equation 11, was developed to predict the elastic moduli of solids with oriented porosity. When applied to solids with spherical pores, it is more accurate than existing equations up to a pore volume fraction of 0.2. At higher volume fractions, it predicts lower (more conservative) values than existing experimental data. Ishai and Cohen's equation [8] is also reasonably accurate up to a pore volume fraction of 0.2, but predicts too high modulus values at higher volume fractions.

When applied to circular cylindrical pores, the proposed equation is closer to available experimental data than that predicted by existing theories [18, 19], especially at higher volume fractions.

Suggestions for further research include extending these same concepts to the prediction of Poisson's ratio and the shear modulus. Also the effect of porosity on tensile strength could be investigated by combining the present approach with that of Stevenson and Ghosh [21].

Acknowledgement

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